

## ORDINARY REDUCTIONS OF ABELIAN VARIETIES

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ABSTRACT. I show that a conjecture of Joshi-Rajan on primes of Hodge-Witt reduction and in particular a conjecture of Jean-Pierre Serre on primes of good, ordinary reduction for an abelian variety over a number field follows from a certain conjecture on Galois representations which may perhaps be easier to prove (and I prove this conjecture for abelian compatible systems of a suitable type). This reduction (to a conjecture about certain systems of Galois representations) is based on a new slope estimate for non Hodge-Witt abelian varieties. In particular for any abelian variety over a number field with at least one prime of good ordinary or split toric reduction, I show that the conjecture of Joshi-Rajan and the conjecture of Serre on ordinary reductions can be reduced to proving that a certain rational trace of Frobenius is in fact an integer. The assertion that this trace is an integer is proved for abelian systems of Galois representations (of suitable type).

It don't mean a thing,  
if it ain't got that swing...

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Louis Armstrong and Duke Ellington  
(and Irving Mills)

## 1. INTRODUCTION

Let  $k$  be a perfect field. Let  $X/k$  be a smooth, projective variety over  $k$ . Let  $W = W(k)$  be the ring of Witt vectors of  $k$ . Let us say, following [Illusie, 1979] that  $X$  is *Hodge-Witt* if the de Rham-Witt cohomology groups  $H^i(X, W\Omega_X^j)$  are finitely generated as  $W$ -modules for all  $i, j \geq 0$ . (see [Illusie, 1979]). Let us say that  $X$  is *ordinary* if and only if  $H^i(X, B\Omega_X^j) = 0$  for all  $i, j \geq 0$ , where  $B\Omega_X^j = d(\Omega_X^{j-1})$  is the sheaf of locally exact  $j$ -forms on  $X$  (see [Illusie, 1979]). It is a theorem of [Illusie and Raynaud, 1983] that if  $X$  is ordinary then  $X$  is Hodge-Witt. For examples of ordinary varieties see [Illusie, 1990].

Suppose now that  $X$  is an abelian variety. Then  $X$  is ordinary if and only if  $X$  has  $p$ -rank equal to  $\dim(X)$ ; and it is a less well-known theorem of Torsten Ekedahl (see [Illusie, 1983]) that  $X$  is Hodge-Witt if and only if  $p$ -rank of  $X$  is at least  $\dim(X) - 1$ . In particular one sees from this that Hodge-Witt but non-ordinary abelian varieties exist.

Readers unfamiliar with [Illusie, 1979], [Illusie and Raynaud, 1983], [Ekedahl, 1985, 1986] should consult Section 2 (especially Remark 2.1) for a “working definition” of Hodge-Witt varieties adapted to abelian varieties which is more than adequate for reading this paper.

Now suppose  $K/\mathbb{Q}$  is a number field and that  $X/K$  is a smooth, projective variety over  $K$ . Then Jean-Pierre Serre has conjectured that  $X$  has ordinary reduction modulo infinitely many non-archimedean primes of  $K$ . In [Joshi and Rajan, 2000] it was conjectured that there exist infinitely many primes at which  $X$  has Hodge-Witt reduction. Clearly if  $X$  has good ordinary reduction at a finite prime  $\mathfrak{p}$  of  $K$  then  $X$  has good, Hodge-Witt reduction at  $\mathfrak{p}$ . Thus the set of primes of good ordinary reduction for  $X$  are contained in the set of primes

of good, Hodge-Witt reduction for  $X$ . In [Joshi, 2014, Theorem 4.1.3] I give, amongst other results, several examples which show that the two sets of primes can have different densities.

In [Joshi, 2014, Theorem 4.1.3] I also showed that Serre's ordinarity conjecture for  $X$  is equivalent to the conjecture on Hodge-Witt reductions due to Joshi-Rajan (also see [Joshi and Rajan, 2000]) for  $X \times_K X$  (this uses an important result of [Ekedahl, 1985]) and I also proved both the conjectures for abelian varieties with complex multiplication (CM case).

In Theorem 4.2 of this note I prove, by using methods quite different from those pursued in [Joshi, 2014, Theorem 4.1.3] for the CM case, that a certain conjecture on Galois representations (see Conjecture 3.6) implies that there exist infinitely many primes of good, Hodge-Witt reductions for any abelian variety. By [Joshi, 2014, Theorem 4.1.3] (stated here as Theorem 4.1) this is enough to prove, assuming Conjecture 3.6, the result conjectured by Serre that any abelian variety over a number field has infinitely many primes of good ordinary reduction (see Theorem 4.6).

Let me remind the reader that for abelian surfaces the result was proved unconditionally by [Ogus, 1982]. In [Joshi and Rajan, 2000], Rajan and I showed, again unconditionally, that there exist infinitely many primes of Hodge-Witt reduction for abelian threefolds. Serre's conjecture on ordinary reductions for abelian varieties has also been established in a few cases under restrictive assumptions on dimensions and endomorphism algebras or Mumford-Tate groups in [Pink, 1998, Tankeev, 1999, Noot, 2000] and the references therein for additional results. The methods of this paper have little overlap with these works.

The proof given here proceeds in two steps. I use an observation of [Joshi, 2014, Theorem 4.1.3] (recalled here as Theorem 4.1) which allows one to reduce to proving that  $A \times A$  has infinitely many primes of Hodge-Witt reduction (see Theorem 4.1). Then I prove, assuming Conjecture 3.6, the existence of infinitely many primes of Hodge-Witt reduction for any abelian variety (see Theorem 4.2). Finally one gets from this, assuming Conjecture 3.6, the ordinarity result conjectured by Serre (see Theorem 4.6). The key new tool in the proofs of these two results, apart from reduction to the Hodge-Witt case provided by [Joshi, 2014, Theorem 4.1.3], is a (sharp) slope estimate (see Theorem 2.2 and Remark 2.24) for non-Hodge-Witt abelian varieties. In Theorem 3.8, I prove Conjecture 3.6 for abelian systems of Galois representations.

In Theorem 5.1 I show that any abelian variety with at least one prime of good ordinary or split toric reduction over a number field has infinitely many primes of Hodge-Witt reduction and hence any such abelian variety also has infinitely many primes of ordinary reduction provided that Conjecture 3.10, which reduces the proof of Conjecture 3.6 to proving that a certain (rational) trace of Frobenius is in fact an integer, is true. In Theorem 3.11 I prove Conjecture 3.10 for abelian systems of Galois representations.

I thank Bryden Cais for many conversations and especially for patiently listening to several of my unsuccessful attempts to prove Conjecture 3.6 in all generality. I also thank Adrian Vasiu for a careful reading of an earlier version of this manuscript and a number of suggestions which led to several improvements and readability of this manuscript. Thanks are also due to Brian Conrad for some conversations on abelian Galois representations.

## 2. NON HODGE-WITT ABELIAN VARIETIES

Let  $X$  be a smooth, projective variety over a perfect field  $k$  of characteristic  $p > 0$ . Let  $W = W(k)$  be the ring of Witt vectors of  $k$  and let its quotient field be  $K_0$ . One says,

following [Illusie and Raynaud, 1983], that  $X$  is Hodge-Witt if and only if the de Rham-Witt cohomology groups  $H^i(X, W\Omega_X^j)$  are of finite type  $W$ -modules for all  $i, j \geq 0$ . Suppose now that  $X$  is an abelian variety. It is a theorem of Torsten Ekedahl (see [Illusie, 1983]) that  $X$  is Hodge-Witt if and only if  $p$ -rank of  $X$  is  $\geq \dim(X) - 1$ .

**Remark 2.1.** Readers unfamiliar with Hodge-Witt varieties may take this  $p$ -rank condition as an *ad hoc* definition of Hodge-Witt abelian varieties. In other words readers may adopt as a “working definition” the statement that  $X$  is Hodge-Witt if and only if  $p$ -rank of  $X$  is  $\geq \dim(X) - 1$ . Since  $p$ -rank of an abelian variety is additive on taking products of abelian varieties, it follows that if  $X, Y$  are abelian varieties over  $k$  then  $X \times Y$  is Hodge-Witt if and only if one of  $X, Y$  is Hodge-Witt and the other is ordinary (this is a very special case of a beautiful general theorem of [Ekedahl, 1985, 1986]). This observation for abelian varieties is adequate for reading this paper. In [Chai et al., 2014] abelian varieties  $X$  which have  $p$ -rank  $\geq \dim(X) - 1$  are called *almost ordinary* and  $p$ -divisible groups arising from them are said to be of *extended Lubin-Tate type*. It should be noted however that neither of these labels reveal the most important property of these abelian varieties: the finiteness of the de Rham-Witt cohomology equivalently the degeneration of the slope spectral sequence at  $E_1$  (see [Illusie and Raynaud, 1983]).

In this section I prove the following slope estimate. As is conventional, a  $p$ -valuation  $v$  used for computing slopes is normalized so that  $v(p) = 1$ .

**Theorem 2.2.** *Let  $X$  be an abelian variety over a perfect field  $k$  of characteristic  $p > 0$ . Assume that  $\dim(X) = g$  and that  $X$  is not Hodge-Witt. Then every slope  $\tilde{\lambda}$  of  $H_{\text{cris}}^g(X/W)$  satisfies*

$$\tilde{\lambda} \geq 1.$$

*Proof.* Let  $M = H_{\text{cris}}^1(X/W) \otimes_W K_0$ , for a slope  $\lambda$  of  $M$ , let  $M_\lambda$  be the slope  $\lambda$  part of  $M$  and in particular let  $M_0, M_1, M_{\frac{1}{2}}$  be respectively the slope zero, slope one and the slope half submodules of  $M$ . Let  $m_\lambda, m_0, m_1, m$  be their dimensions respectively. Further as  $X$  is not Hodge-Witt so

$$(2.3) \quad m_0 \leq g - 2,$$

and one has  $\dim_{K_0}(M) = 2g$ . So one gets

$$(2.4) \quad 2g = \dim(M_0) + \dim(M_1) + \dim(M_{\frac{1}{2}}) + \sum_{\lambda \neq 0, \frac{1}{2}, 1} \dim(M_\lambda).$$

By definition of  $p$ -rank, the  $p$ -rank of  $X$  is  $m_0$ . By duality for abelian varieties  $m_\lambda = m_{1-\lambda}$ . Therefore one can write this as

$$(2.5) \quad 2g = 2m_0 + m + 2 \sum_{0 < \lambda < \frac{1}{2}} m_\lambda.$$

Hence one has, for every slope  $0 < \lambda < \frac{1}{2}$  of  $M$ ,

$$(2.6) \quad 2m_\lambda \leq 2g - 2m_0 - m.$$

Observe that  $g - m_0 - \frac{m}{2}$  cannot be negative by (2.5). Now the proof is split into two cases according to whether  $g - m_0 - \frac{m}{2} \neq 0$  or  $g - m_0 - \frac{m}{2} = 0$ .

First suppose  $g - m_0 - \frac{m}{2} \neq 0$ . So  $m_\lambda \leq g - m_0 - \frac{m}{2}$ . Now for  $0 < \lambda < 1$  let  $\lambda = \frac{a}{b}$  with  $a \geq 1, b > 1$  and  $(a, b) = 1$ . Then  $b|m_\lambda$  so  $b \leq m_\lambda$  and hence  $\frac{1}{b} \geq \frac{1}{m_\lambda}$ . Hence in particular for any  $0 < \lambda < \frac{1}{2}$  one has

$$(2.7) \quad \lambda \geq \frac{1}{m_\lambda}.$$

As  $m_\lambda \leq g - m_0 - \frac{m}{2}$ , from (2.6) and from (2.7), for any  $0 < \lambda < \frac{1}{2}$ , one has the fundamental estimate:

$$(2.8) \quad \lambda \geq \frac{1}{m_\lambda} \geq \frac{1}{g - m_0 - \frac{m}{2}}.$$

If  $0 < \lambda < \frac{1}{2}$  is a slope, then so is  $1 - \lambda$  and  $1 > 1 - \lambda > \frac{1}{2}$  and one has

$$(2.9) \quad 1 - \lambda > \lambda \geq \frac{1}{g - m_0 - \frac{m}{2}},$$

and hence one has for all slopes  $\lambda \neq 0, 1, \frac{1}{2}$  the estimate:

$$(2.10) \quad \lambda \geq \frac{1}{g - m_0 - \frac{m}{2}}.$$

Now recall that  $H_{\text{cris}}^g(X/W) \otimes K_0 = \wedge^g H_{\text{cris}}^1(X/W) = \wedge^g M$  and so any slope  $\tilde{\lambda}$  of Frobenius on  $H_{\text{cris}}^g$  may be computed from slopes of  $M$ . Any slope  $\tilde{\lambda}$  of  $H_{\text{cris}}^g$  is of the form

$$(2.11) \quad \tilde{\lambda} = \lambda_1 + \cdots + \lambda_g$$

for some slopes  $\lambda_1, \lambda_2, \dots, \lambda_g$  of  $M$ . If any of the  $\lambda_j$  occurring in this expression is equal to one, then  $\tilde{\lambda} \geq 1$  holds trivially and there is nothing to prove. So assume  $\lambda_j < 1$  for all  $j = 1, \dots, g$ . Let  $i_0$  be the number of times  $\lambda_j = 0$  and let  $i$  be the number of times  $\frac{1}{2}$  occurs in the above representation. Then

$$(2.12) \quad \tilde{\lambda} \geq \frac{i}{2} + (g - i_0 - i) \left( \frac{1}{g - m_0 - \frac{m}{2}} \right)$$

$$(2.13) \quad \geq \left( \frac{g - i_0}{g - m_0 - \frac{m}{2}} \right) + i \left( \frac{1}{2} - \frac{1}{g - m_0 - \frac{m}{2}} \right).$$

Note that  $\frac{1}{2} - \frac{1}{g - m_0 - \frac{m}{2}} > 0$ . This follows from (2.5) as

$$(2.14) \quad g - m_0 - \frac{m}{2} = \sum_{0 < \lambda < \frac{1}{2}} m_\lambda$$

and for slopes  $0 < \lambda < \frac{1}{2}$  one has  $m_\lambda \geq 3$  which gives

$$(2.15) \quad g - m_0 - \frac{m}{2} \geq 3,$$

and hence

$$(2.16) \quad \frac{1}{g - m_0 - \frac{m}{2}} \leq \frac{1}{3}.$$

So  $\frac{1}{2} > \frac{1}{3} \geq \frac{1}{g - m_0 - \frac{m}{2}}$ . Thus the second term on the right hand side of (2.12) is positive. Now as

$$(2.17) \quad i_0 \leq m_0$$

so

$$(2.18) \quad g - i_0 \geq g - m_0 \geq g - m_0 - \frac{m}{2}.$$

Thus in the case  $g - m_0 - \frac{m}{2} > 0$  one has

$$(2.19) \quad \tilde{\lambda} \geq \frac{g - i_0}{g - m_0 - \frac{m}{2}} \geq \frac{g - m_0}{g - m_0 - \frac{m}{2}} \geq 1.$$

Let me now address the case  $g - m_0 - \frac{m}{2} = 0$ . One argues directly using (2.5) from which one sees that  $m_\lambda = 0$  for all  $\lambda \neq 0, \frac{1}{2}, 1$ . From

$$(2.20) \quad \tilde{\lambda} = \lambda_1 + \cdots + \lambda_g$$

assuming once again that none of the slopes in the above expression for  $\tilde{\lambda}$  are equal to one, one has the estimate

$$(2.21) \quad \tilde{\lambda} = \frac{1}{2}(g - i_0) \geq \frac{1}{2}(g - m_0) = \frac{m}{4}.$$

So to prove our claim that  $\tilde{\lambda} \geq 1$  it suffices to show that if  $g - m_0 - \frac{m}{2} = 0$  then  $m \geq 4$ . If  $m < 4$  then as  $2|m$  this says  $m = 2$  or  $m = 0$ . If  $m = 2$  then

$$(2.22) \quad g = m_0 + \frac{m}{2} = m_0 + \frac{2}{2} = m_0 + 1$$

and hence  $m_0 = g - 1$  and hence  $X$  is Hodge-Witt. This contradicts our assumption that  $X$  is non Hodge-Witt. Similarly if  $m = 0$  then  $g = m_0$  hence  $X$  is ordinary but this contradicts our assumption that  $X$  is non Hodge-Witt. Thus if  $g - m_0 - \frac{m}{2} = 0$  then  $m \geq 4$ ,  $0 \leq m_0 \leq g - 2$  and  $\tilde{\lambda} = \frac{m}{4} \geq 1$ . Thus one has in all cases

$$(2.23) \quad \tilde{\lambda} \geq 1.$$

This completes the proof of the theorem.  $\square$

**Remark 2.24.** Suppose  $X$  is ordinary. Then  $H_{cris}^g(X/W)$  has a slope zero part of rank one. So  $\tilde{\lambda} \geq 1$  cannot hold for all  $\tilde{\lambda}$ . Similarly if  $X$  is Hodge-Witt, but not ordinary, then  $X$  has  $p$ -rank  $g - 1$  hence there is a slope  $\tilde{\lambda} = \frac{1}{2}$  for  $H_{cris}^g(X/W)$ . So the assumption that  $X$  is non Hodge-Witt cannot be relaxed to  $X$  is non-ordinary in Theorem 2.2. Moreover for every  $g \geq 2$  there exists, by [Tate, 1968-1969], an abelian variety over an algebraically closed field  $k$  of characteristic  $p > 0$  with slopes  $\{\frac{1}{g}, 1 - \frac{1}{g}\}$  each with multiplicity  $g$  and this abelian variety is manifestly non Hodge-Witt with exactly one slope  $\tilde{\lambda}$  with  $\tilde{\lambda} = 1$ . So the estimate is the best possible in all dimensions  $g \geq 2$ .

### 3. A CONJECTURE ABOUT CERTAIN GALOIS REPRESENTATIONS

I make the following definition. Let  $K$  be a finite extension of  $\mathbb{Q}$ , and fix an algebraic closure of  $K$  and let  $G_K$  be the Galois group of  $K$  (with respect to this algebraic closure). For any finite set of primes  $S$  of  $K$ , and a rational prime  $p$ , let  $S_\ell = S \cup \{\ell : \ell | \ell\}$ . For a  $p$ -adic representation of  $G_K$  and prime  $\mathfrak{p} | p$  of  $K$ , I will write  $V_{\mathfrak{p}}$  for the restriction of the  $G_K$ -representation  $V_p$  to the decomposition group  $D_{\mathfrak{p}}$  at  $\mathfrak{p}$ . Let  $D_{cris}(V_{\mathfrak{p}})$  be the (covariant) functor constructed in [Fontaine, 1994]. For any prime  $\ell$ , let  $\mathbb{Q}_\ell(-1)$  be the  $\ell$ -adic Tate twist. This is a  $G_K$ -representation of Weil weight two and the unique slope of Frobenius on  $D_{cris}(\mathbb{Q}_p(-1))$  is one. With this convention  $\mathbb{Q}_\ell(1)$  has Weil-weight minus two.

**Definition 3.1.** Let  $g \geq 2$  be an integer. Let  $S$  be a finite set of primes of  $K$ . A pair of continuous,  $\ell$ -adic Galois representations (one for each rational prime  $\ell$ ),  $\{V_\ell\}_\ell, \{V'_\ell\}_\ell$  of  $G_K$  are said to be a *twist-coupled system of Galois representations of type  $(g, g-2)$*  if they satisfy the following conditions:

- (G.1) Each  $V_\ell$  is unramified outside  $S \cup \{\ell\}$ .
- (G.2) Each  $V_\ell$  is pure of Weil weight  $g$ .
- (G.3) For any prime  $\mathfrak{p} \notin S_\ell$  the characteristic polynomial of Frobenius of  $V_\ell$  at  $\mathfrak{p}$  is a polynomial  $f_{\mathfrak{p},\ell}(t) \in \mathbb{Q}[t]$ , and if  $\ell, \ell'$  are primes and  $\mathfrak{p} \notin S_\ell \cup S_{\ell'}$  then  $f_{\mathfrak{p},\ell}(t) = f_{\mathfrak{p},\ell'}(t)$ .
- (L.1) For all primes  $\mathfrak{p}$ ,  $V_{\mathfrak{p}}$  is potentially semistable.
- (L.2) For all primes  $\mathfrak{p} \notin S$ ,  $V_{\mathfrak{p}}$  is crystalline.
- (L.3) For all primes  $\mathfrak{p} \notin S$  the characteristic polynomial of the linearized Frobenius of  $D_{\text{cris}}(V_{\mathfrak{p}})$  is  $f_{\mathfrak{p}}(t) \in \mathbb{Q}[t]$  and if  $\mathfrak{p} \notin S_\ell$  one has  $f_{\mathfrak{p},\ell}(t) = f_{\mathfrak{p}}(t)$ .
- (E) Except for a set of finitely many primes  $\mathfrak{p}$  including those in  $S$ , the characteristic polynomial  $f_{\mathfrak{p}}(t) \in \mathbb{Z}[t]$ .
- (S) For all primes outside the exceptional set of primes of (E) the slopes of Frobenius on  $D_{\text{cris}}(V_{\mathfrak{p}})$  are in the interval  $[1, g-1]$ .
- (H) For all primes  $\mathfrak{p}$ ,  $V_{\mathfrak{p}}$  has Hodge-Tate weights in  $[0, g]$ , and one has  $\text{gr}^0(D_{HT}(V_{\mathfrak{p}})) \neq 0$ .
- (T) For all  $\ell$  there exists a continuous isomorphism  $V_\ell \xrightarrow{\sim} V'_\ell(-1)$  of  $G_K$ -modules (here  $\mathbb{Q}_\ell(-1)$  is the Tate twist).

The following simple lemma, while not essential in my proof, explicates the condition (S).

**Lemma 3.2.** *Let  $f(t) \in \mathbb{Z}[t]$  be a non-zero polynomial whose roots are  $p$ -Weil numbers of weight  $m \geq 1$ . Suppose that  $L/\mathbb{Q}$  is a finite extension containing all the roots of  $f(t)$  and suppose for any  $p$ -adic valuation  $v_p$  of  $L$ , normalized so that  $v_p(p) = 1$ , extending the standard  $p$ -adic valuation of  $\mathbb{Q}_p$ , one has  $v_p(\alpha) \geq 1$  for any root  $\alpha$  of  $f(t)$ . Then there is an algebraic integer  $\beta$  such that  $\alpha = p\beta$ .*

*Proof.* Let  $(p) = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r}$  be the prime factorization of  $p$  in  $L$ . Then the estimate  $v_{\mathfrak{p}_i}(\alpha) \geq 1$ , with the normalization  $v_{\mathfrak{p}_i}(p) = 1$ , says that  $\alpha \in \mathfrak{p}_i^{a_i}$ . Hence  $\alpha \in \mathfrak{p}_i^{a_i}$  for all  $i = 1, \dots, r$ . Thus  $\alpha \in \cap \mathfrak{p}_i^{a_i} = \mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_r^{a_r} = (p)$ . Therefore  $\alpha = p\beta$  for some algebraic integer  $\beta$  as asserted.  $\square$

I will write the quantities corresponding to  $V'_p$ , such as characteristic polynomials of Frobenius, traces of Frobenius etc. as primed quantities:  $f'_p(t), f'_{\mathfrak{p},\ell}(t)$  etc. Hopefully there will be no confusion with derivatives (which will not be used in this paper).

The contents of the following remarks will clarify this list of properties of twist-coupled systems and will be used in the rest of the paper. The remarks are immediate from definitions or elementary considerations.

**Remark 3.3.** Note that the properties are indexed by global conditions (G.1)–(G.3) and by local conditions (L.1)–(L.3) which members of  $\{V_\ell\}_\ell$  satisfy. The assumptions (G.1)–(G.3) and (L.1)–(L.3) are satisfied by all geometric Galois representations. That (G.1)–(G.3), (L.1)–(L.3) hold in geometric contexts is the work of many mathematicians: properties (G.1)–(G.3) hold by [Deligne, 1974], properties (L.1)–(L.3) first arose in the work of [Fontaine, 1982] and have now been established in geometric contexts (see [Colmez, 2001–2002] for a detailed bibliography). That the property (L.3) holds in geometric context is due to [Katz and Messing, 1974].

- (1) Through (T) the representation  $\{V'_\ell\}_\ell$  also satisfies (G.1)–(G.3) and (L.1)–(L.3).



- (2) Properties **(E)**, **(S)**, **(H)** are not invariant under Tate twists and hence are not inherited by  $\{V'_\ell\}$  through **(T)**.
- (3) Property **(E)** of  $V_p$  says that in some sense  $V_p$  is “effective.”
- (4) Note that properties **(S)**, **(E)** and **(H)** only involve  $\{V_\ell\}_\ell$ .

**Proposition 3.4.** *Suppose  $\{V_\ell, V'_\ell\}_\ell$  is a twist-coupled family of Galois representations. Then*

- (1) *Properties **(G.1)–(G.3)**, **(L.1)–(L.3)** are hold for  $\{V'_\ell\}$ .*
- (2) *The property **(E)** is inherited by  $\{V'_\ell\}$ .*
- (3) *For all primes  $\mathfrak{p} \notin S$ ,  $V'_\mathfrak{p}$  is crystalline.*
- (4) *For all but a finite number of primes, the slopes of Frobenius on  $D_{\text{cris}}(V'_\mathfrak{p})$  are in  $[0, g - 2]$ .*
- (5) *The family  $\{V_\ell^{ss}, V_\ell'^{ss}\}_\ell$  of semisimplifications of  $\{V_\ell, V'_\ell\}_\ell$  is also a twist coupled system.*

*Proof.* (1), (3) and (4) are immediate from the fact that **(G.1)–(G.3)**, **(L.1)–(L.3)**, **(S)**, **(E)**, **(H)** and **(T)** hold for  $\{V_\ell\}$  and as  $V'_\ell = V_\ell(1)$  by **(T)**. Claim (2) is now immediate from **(T)**, **(E)** and **(S)** for  $\{V_\ell\}$ . Clearly (5) is entirely formal.  $\square$

**Remark 3.5.** In particular one sees from Proposition 3.4(2) that if **(G.1)–(G.3)**, **(L.1)–(L.3)**, **(S)**, **(E)**, **(H)** and **(T)** hold then **(E)** also holds for  $\{V'_\ell\}_\ell$ . As  $V_\ell = V'_\ell(-1)$  by **(T)**, and as **(E)** holds for  $V'_\ell$ , one has *divisibility of traces of Frobenius elements acting on  $V_\ell$* . Thus the conditions **(G.1)–(G.3)**, **(L.1)–(L.3)**, **(S)**, **(E)**, **(H)** and **(T)** provide a natural way of encoding the divisibility by  $p$  of trace of Frobenius element at  $\mathfrak{p}$  acting on  $V_\ell$ .

I propose the following conjecture.

**Conjecture 3.6.** Let  $X/K$  be a smooth, projective variety over a number field  $K$  of dimension  $g \geq 2$  with

$$H^g(X, \mathcal{O}_X) \neq 0.$$

Then the family of  $\ell$ -adic Galois representations:

$$(3.7) \quad \{V_\ell = H_{\text{et}}^g(X, \mathbb{Q}_\ell), V'_\ell = V_\ell(1)\}_\ell$$

is not a twist coupled system of Galois representations of type  $(g, g - 2)$ .

For  $g = 2$  and any smooth projective variety  $X$  (with  $H^2(X, \mathcal{O}_X) \neq 0$ ), Conjecture 3.6 is immediate by using methods of [Ogus, 1982, Joshi and Rajan, 2000, Bogomolov and Zarhin, 2009]. To see this it is sufficient to note that the methods of *loc. cit.* show that if the traces of Frobenius at all but finite number of primes  $p$  (for  $\ell$ -adic étale cohomology, with an appropriate  $\ell$  for  $X$ ) are divisible by  $p$ , then the semisimplification of  $H_{\text{et}}^2(X, \mathbb{Q}_\ell)$  is isomorphic to  $\oplus \mathbb{Q}_\ell(-1)$  (as a  $G_K$ -module) (and by compatibility (**(G.1)–(G.3)**) this must hold for all  $\ell$ ) which obviously contradicts **(H)** (by the Hodge-Tate decomposition Theorem furnished by  $p$ -adic Hodge Theory). Thus  $\{V_\ell = H_{\text{et}}^2(X, \mathbb{Q}_\ell), V'_\ell = V_\ell(1)\}_\ell$  is not a twist-coupled system. Note that the hypothesis in *loc. cit.* for  $g = 2$ , that  $X$  is a K3 surface is used only to prove that there is exactly one unit eigenvalue of Frobenius. Here is some additional evidence for Conjecture 3.6.

**Theorem 3.8.** *Let  $X$  be a smooth, projective variety of dimension  $g$  over a number field and suppose  $H^g(X, \mathcal{O}_X) \neq 0$ . Assume that  $\{(\rho_\ell, H_{\text{et}}^g(X, \mathbb{Q}_\ell))\}_\ell$  is a family of abelian Galois representations (i.e. factors as a representation  $\rho_\ell : G_K^{\text{ab}} \rightarrow \text{GL}(H_{\text{et}}^g(X, \mathbb{Q}_\ell))$ ). Then Conjecture 3.6 holds for  $X$ .*

*Proof.* As  $\{H_{et}^g(X, \mathbb{Q}_\ell)\}_\ell$  is a family of abelian representations, so is its Tate-twist  $\{V'_\ell = H_{et}^g(X, \mathbb{Q}_\ell)(1)\}_\ell$ . Replacing these representations by their semi-simplifications one can assume that one has a family of semi-simple abelian representations. Assume the Conjecture 3.6 is false for  $X$ . This means that

$$(3.9) \quad \{V_\ell = H_{et}^g(X, \mathbb{Q}_\ell), H_{et}^g(X, \mathbb{Q}_\ell)(1)\}_\ell$$

forms a twist-coupled system of  $G_K$ -representations (so properties (G.1)–(G.3), (L.1)–(L.3), (S), (E), (H) and (T) hold). In particular by Proposition 3.4(2),  $\{V'_\ell\}$  satisfies (E). By (G.1)–(G.3), (L.1)–(L.3) and [Serre, 1968, Theorem of Tate, page III-7]  $V'_p$  is locally algebraic and hence by [Serre, 1968, Proposition, page III-9]  $V'_p$  arises from a  $\mathbb{Q}$ -rational representation of an algebraic torus. It is possible to choose a prime  $p$  such that for any prime  $\mathfrak{p}$  of  $K$  lying over  $p$  such that  $V'_p$  is crystalline at  $\mathfrak{p}$  (and hence Hodge-Tate at  $\mathfrak{p}|p$ ) and the algebraic torus giving rise to this representation is split at  $p$ . Pick such a prime  $p$ . Then  $V'_p$  is a direct sum of crystalline characters of  $D_{\mathfrak{p}}$ . As  $\{V'_\ell\}$  satisfies (E), the characteristic polynomials of Frobenius elements at all but finite number of primes of  $K$  (acting on  $V'_p$ ) have integer coefficients. Then the integrality of the characteristic polynomials of Frobenius elements in  $G_K$  (over all but finitely many primes of  $K$ ) shows by [Serre, 1968, Chapter II, Corollary 2, page II-36] that the Hodge-Tate weights of  $V'_p$  are non-negative (note that *loc. cit.* the Hodge-Tate weights are determined by the algebraic characters of the aforementioned torus which appear in rational representation of this torus provided by [Serre, 1968, Proposition, page III-9]). So  $V'_p$  has non-negative Hodge-Tate weights and hence  $V_p = V'_p(-1)$  has strictly positive Hodge-Tate weights. This contradicts (H) as a Theorem of Tate (see [Tate, 1967]) says that if  $V_p$  is a  $p$ -adic representation with strictly positive Hodge-Tate weights (at primes over  $p$ ) then  $(V_p \otimes \mathbb{C}_p)^{D_{\mathfrak{p}}} = 0$  for any prime  $\mathfrak{p}$  lying over  $p$ . On the other hand one has  $H^g(X, \mathcal{O}_X) \neq 0$  and hence  $(V_p \otimes \mathbb{C}_p)^{D_{\mathfrak{p}}} \neq 0$ . Hence one has arrived at a contradiction.  $\square$

In some situations one can replace Conjecture 3.6 by the somewhat simpler Conjecture 3.10 given below.

**Conjecture 3.10.** Let  $K$  be a number field. Let  $S$  be a finite set of primes of  $K$ . Suppose  $\{V'_\ell\}_\ell$  is a family of continuous Galois representations satisfying the properties (G.1)–(G.3), (L.1)–(L.3), (E) and suppose that for a prime  $\mathfrak{p}$  (lying over a rational prime  $p$ ) the following hypothesis hold (here  $D_{\mathfrak{p}} \subset G_K$  is the decomposition group at  $\mathfrak{p}$  and  $I_{\mathfrak{p}}$  is the inertia subgroup at  $\mathfrak{p}$ , and  $V'_p$  is the restriction of  $V'_p$  to  $D_{\mathfrak{p}}$ ):

- (O.1) The vector of Hodge-Tate weights of  $V'_q$  is constant as  $q$  varies over primes of  $K$ .
- (O.2) For any prime  $\ell$  not lying below  $\mathfrak{p}$ , the representation  $V'_\ell$  of  $D_{\mathfrak{p}}$  is unramified (resp. unipotent).
- (O.3) The representation  $V'_p$  of  $D_{\mathfrak{p}}$  is crystalline ordinary (resp. semi-stable ordinary, i.e., equipped with a  $D_{\mathfrak{p}}$ -invariant filtration whose graded pieces are isomorphic to  $\mathbb{Q}_p(-i)$  for  $i \in \mathbb{Z}$ ).
- (O.4) For any prime  $\ell$  not lying below  $\mathfrak{p}$ , the characteristic polynomial of Frobenius at  $\mathfrak{p}$  acting on  $V'_\ell$  coincides with the characteristic polynomial of Frobenius on  $D_{st}(V'_p)$ .
- (O.5) The trace  $a'_p$  of Frobenius  $\phi'_p$  on  $D_{st}(V'_p)$  is rational (i.e.  $a'_p \in \mathbb{Q}$ ).

Then the trace  $a'_p$  of Frobenius  $\phi'_p$  on  $D_{st}(V'_p)$  is an integer.

Here is some partial evidence for Conjecture 3.10.



**Theorem 3.11.** *Suppose  $\{\rho'_\ell : G_K^{ab} \rightarrow \mathrm{GL}(V'_\ell)\}_\ell$  is an family of continuous, abelian Galois representations which satisfies all the hypothesis of Conjecture 3.10. Then Conjecture 3.10 is true for  $\{V'_\ell\}_\ell$ .*

*Proof.* This is proved in a manner similar to Theorem 3.8. Replacing our system by its semisimplification one can assume that one has a semisimple system of representations of  $G_K^{ab}$ . First choose a prime  $q \neq p$  and a prime  $\mathfrak{q}$  of  $K$  lying over  $q$  such that  $V'_\mathfrak{q}$  is crystalline at  $\mathfrak{q}$ . This means, by (G.1)–(G.3), (L.1)–(L.3) and [Serre, 1968, Theorem of Tate, page III-7], that  $V'_\mathfrak{q}$  is locally algebraic and hence by [Serre, 1968, Proposition, page III-9]  $V'_\mathfrak{q}$  arises from a  $\mathbb{Q}$ -rational representation of an algebraic torus. As  $\{V'_\ell\}$  satisfies (E), the characteristic polynomials of Frobenius elements at all but finite number of primes (acting on  $V'_\mathfrak{q}$ ) are integers. The integrality of the characteristic polynomials of Frobenius elements in  $G_K$  (over all but finitely many primes of  $K$ ) shows, by [Serre, 1968, Chapter II, Corollary 2, page II-36], that the Hodge-Tate weights of  $V'_\mathfrak{q}$  are non-negative. By (O.1) the Hodge-Tate weights of  $V'_\mathfrak{p}$  are also non-negative. Since  $V'_\mathfrak{p}$  is semistable and ordinary (by (O.3)), the non-negativity of Hodge-Tate weights of  $V'_\mathfrak{p}$  says that the characteristic polynomial of  $\phi'_\mathfrak{p}$  on  $D_{st}(V'_\mathfrak{p})$  has  $p$ -adic integer coefficients. Thus by (O.5) one sees that the trace of Frobenius  $\phi'_\mathfrak{p}$  is a rational integer. This proves the assertion.  $\square$

#### 4. HODGE-WITT AND ORDINARY REDUCTIONS

Before proceeding further let me recall the following observation of [Joshi, 2014, Theorem 4.1.3]. The only case Theorem 4.1 of interest for this paper is the case when  $X$  is an abelian variety and as remarked in Remark 2.1 if one uses the “working definition” that an abelian variety  $X$  is Hodge-Witt if and only if  $X$  has  $p$ -rank  $\geq \dim(X) - 1$ , then Theorem 4.1 given below is quite elementary to prove.

**Theorem 4.1.** *Let  $X$  be a smooth, projective variety over a number field  $K$ . Then the following are equivalent:*

- (1) *There exist infinitely many primes of ordinary reduction for  $X$ .*
- (2) *There exist infinitely many primes of ordinary reduction for  $X \times_K X$*
- (3) *There exist infinitely many primes of Hodge-Witt reduction for  $X \times_K X$ .*

For  $\dim(X) = 3$  the existence of infinitely many primes of Hodge-Witt reductions, (without assuming Conjecture 3.6), is due to [Joshi and Rajan, 2000] (also see [Joshi, 2014]).

**Theorem 4.2.** *Let  $K$  be a number field. Let  $X/K$  be an abelian variety over  $K$  of dimension  $g \geq 1$ . Assume Conjecture 3.6 is true for  $X$  if  $g \geq 2$ . Then there exist infinitely many primes of Hodge-Witt reduction for  $X$ .*

*Proof.* Since any abelian variety of dimension  $g = 1$  has Hodge-Witt for an infinite set of primes, one can assume that  $g \geq 2$ . Suppose the assertion is not true. Then I show that

$$(4.3) \quad \{V_\ell = H_{et}^g(X, \mathbb{Q}_\ell), V'_\ell = V_\ell(1)\}_\ell$$

is a system of twist-coupled Galois representations of type  $(g, g-2)$ . Note that (T) holds as one has the tautological isomorphism  $V_\ell = V_\ell(1)(-1) = V'_\ell(-1)$ . Clearly  $\{V_\ell, V'_\ell\}_\ell$  satisfies all the properties (G.1)–(G.3), (L.1)–(L.3), (S), (E), (H) and (T) except possibly (S). By our assumption  $X$  does not have Hodge-Witt reduction at all but a finite number of primes  $\mathfrak{p}$  of  $K$ . Suppose  $\mathfrak{p}$  is a prime of good, non Hodge-Witt reduction. Let  $X_\mathfrak{p}$  be the reduction modulo  $\mathfrak{p}$  of  $X$  (in a regular, smooth, proper model of  $X$  over a suitable localization of the

ring of integers of  $K$ ). Let  $\kappa(\mathfrak{p})$  be the residue field of  $\mathfrak{p}$ . Then by Theorem 2.2 one sees that if  $\alpha$  is any eigenvalue of Frobenius on  $H_{et}^g(X, \mathbb{Q}_\ell)$  (equivalently on  $H_{cris}^g(X_{\mathfrak{p}}/W(\kappa(\mathfrak{p})))$ , by [Katz and Messing, 1974]) then  $\alpha$  satisfies  $v_p(\alpha) \geq 1$ . As  $X_{\mathfrak{p}}$  has dimension  $g$ , one has by Poincaré duality for crystalline cohomology of  $X_{\mathfrak{p}}$  that  $v_p(\alpha) \leq g - 1$ . Hence for any prime  $\mathfrak{p}$  of non Hodge-Witt reduction (S) holds for  $D_{cris}(V_{\mathfrak{p}})$ . Thus one has arrived at a twist coupled system of Galois representations of type  $(g, g - 2)$ . So if Conjecture 3.6 is true for  $X$  then one has arrived at a contradiction.  $\square$

**Corollary 4.4.** *Let  $X/K$  be an abelian variety over a number field  $K$ . If Conjecture 3.6 is true for  $X$  then there exists a set of primes of positive density at which  $X$  has Hodge-Witt reduction.*

*Proof.* If the assertion is not true then the set of primes of Hodge-Witt reduction has density zero and examining the proof of Theorem 4.2 one is again led to a contradiction.  $\square$

**Remark 4.5.** Let me point out that if one assumes that  $X$  has non-ordinary reduction at  $\mathfrak{p}$  in Theorem 2.2 then it follows that the traces of Frobenius of  $\mathfrak{p}$  on  $H_{et}^g(X, \mathbb{Q}_\ell)$  are divisible by  $p$ . However this is not enough to conclude that all eigenvalues of Frobenius are divisible by  $p$ . Thus the assumption in Theorem 2.2 that  $X$  is non Hodge-Witt places stronger constraints on the crystalline cohomology of  $X$  than non-ordinarity does.

**Theorem 4.6** (Serre's Ordinarity Conjecture). *Let  $A/K$  be any abelian variety over  $K$ . If Conjecture 3.6 is true for  $A \times_K A$  then there exists a set of primes of positive density of good ordinary reduction for  $A$ .*

*Proof.* It is sufficient, by [Joshi, 2014, Theorem 4.1.3] (also see Remark 2.1), to prove that  $X = A \times_K A$  has infinitely many primes of Hodge-Witt reduction. This is immediate from Theorem 4.2 if  $A \times_K A$  satisfies Conjecture 3.6. The density assertion follows from Theorem 4.2 and Corollary 4.4.  $\square$

**Corollary 4.7.** *Let  $K$  be any field of characteristic zero. If Conjecture 3.6 is true for an abelian variety and also for the self-product of this variety over  $K$  then this abelian variety has infinitely many primes of Hodge-Witt and ordinary reductions.*

*Proof.* This is immediate from Theorem 4.6.  $\square$

**Remark 4.8.** In [Joshi, 2014] it is conjectured that for a class of smooth, projective varieties including abelian varieties, the set of primes of non Hodge-Witt reductions in dimension  $\dim(X)$  is also infinite. This, I believe, is the correct analog in higher dimensions, of Elkies' Theorem (see [Elkies, 1987]) on the infinitude of primes of supersingular reductions for elliptic curves over  $\mathbb{Q}$  (in [Joshi, 2014] it is noted that Elkies's result is a very special case of this conjecture).

## 5. THE TORIC CASE

In case of abelian varieties over a number field with at least one prime of ordinary or split-toric reduction one may replace Conjecture 3.6 by Conjecture 3.10 in the proof of Theorem 4.2. This is proved in Theorem 5.1 below.

**Theorem 5.1.** *Let  $X/K$  be an abelian variety of dimension  $g \geq 2$  over a number field  $K$  and suppose that  $\mathfrak{p}$  is a prime of  $K$  at which  $X$  has either ordinary or split toric reduction. If Conjecture 3.10 is true then  $X$  has infinitely many primes of Hodge-Witt reduction.*

*Proof.* Let us suppose the theorem is false. Then the proof of Theorem 4.2 shows that  $\{V_\ell = H_{et}^g(X, \mathbb{Q}_\ell), V'_\ell = V_\ell(1)\}_\ell$  forms a twist-coupled system of Galois representations. So Properties (G.1)–(G.3), (L.1)–(L.3), (S), (E), (H) and (T) hold.

Let  $a_{\mathfrak{p}}$  (resp.  $a'_{\mathfrak{p}}$ ) be the trace of Frobenius on  $D_{st}(V_p)$  (resp.  $D_{st}(V'_p)$ ). Suppose for the moment that  $a_{\mathfrak{p}}$  is an integer. Then the property (T) says that  $a'_{\mathfrak{p}}$  is rational. Since  $V_p$  is ordinary and  $H^g(X, \mathcal{O}_X) \neq 0$  one has

$$(5.2) \quad a_{\mathfrak{p}} \not\equiv 0 \pmod{p}.$$

On the other hand the isomorphism  $V_\ell \simeq V'_\ell(-1)$  of  $G_K$ -modules says that one has

$$(5.3) \quad a_{\mathfrak{p}} = a'_{\mathfrak{p}} \cdot p.$$

Now it is clear that (5.2) and (5.3) cannot hold simultaneously provided one knows that  $a'_{\mathfrak{p}} \in \mathbb{Z}$ .

First suppose that  $X$  has good ordinary reduction at  $\mathfrak{p}$ . Suppose  $\ell$  is not a prime lying below  $\mathfrak{p}$ . Then by the hypothesis, that  $X$  has good reduction at  $\mathfrak{p}$ , the trace of Frobenius at  $\mathfrak{p}$  acting on  $V_\ell$  is an integer. By (L.3) this integer is also the trace of Frobenius acting on  $D_{st}(V_p) = D_{crys}(V_p)$ . As  $V_p = V_p(1)$  this says that the trace of Frobenius at  $\mathfrak{p}$  acting on  $V'_\ell$  is rational. Hence all the hypothesis of Conjecture 3.10 hold for  $\{V'_\ell\}_\ell$ . Thus one can invoke Conjecture 3.10 and the assertion follows.

Now suppose that  $\mathfrak{p}$  is a prime of split toric reduction lying over a rational prime  $p$ . Let us consider the restriction of the  $G_K$ -representation of  $V_\ell$  (for a rational prime  $\ell$  not lying below  $\mathfrak{p}$ ) to the decomposition group  $D_{\mathfrak{p}}$  at  $\mathfrak{p}$ . Let  $K_{\mathfrak{p}}$  be the completion of  $K$  at  $\mathfrak{p}$ .

I have to prove that the traces  $a_{\mathfrak{p}}$  (resp.  $a'_{\mathfrak{p}}$ ) are rational. This is certainly well-known but I recall this for convenience.

Let  $\sigma$  be any element of the Weil group of  $K_{\mathfrak{p}}$  (over its algebraic closure) lifting Frobenius of the residue field at  $\mathfrak{p}$  and acting on  $V_\ell$  (through the action of  $D_{\mathfrak{p}}$  on  $V_\ell$ ). By abuse of terminology I will call such an element a Frobenius at  $\mathfrak{p}$ . Then it follows from [Grothendieck, 1972] and [Coleman and Iovita, 2009] that  $a_{\mathfrak{p}} = \text{Tr}(\rho_\ell(\text{Frob}_{\mathfrak{p}}))$  and it is immediate from [Grothendieck, 1972, Theorem 4.3(b), page 359] and the fact that as  $D_{\mathfrak{p}}$ -modules one has an isomorphism  $V_\ell = H_{et}^g(X, \mathbb{Q}_\ell) = \wedge^g H_{et}^1(X, \mathbb{Q}_\ell)$  that  $a_{\mathfrak{p}}$  is an integer and so  $a'_{\mathfrak{p}}$  is rational. Hence all the hypothesis of Conjecture 3.10 hold for  $\{V'_\ell\}_\ell$ . Now one invokes Conjecture 3.10 to see that  $a'_{\mathfrak{p}}$  is an integer. This together with (5.3) and (5.2) completes the proof.  $\square$

This has the following corollary.

**Theorem 5.4.** *Let  $X/K$  be an abelian variety over a number field  $K$  and suppose that  $\mathfrak{p}$  is a prime of  $K$  at which  $X$  has good ordinary or split toric reduction. Assume that Conjecture 3.10 is true. Then  $X$  has infinitely many primes of ordinary reduction.*

*Proof.* This follows from Theorem 5.1 and Theorem 4.1 because if  $X$  has a prime of split toric reduction then so does  $X \times X$ . Similarly if  $X$  has one prime of good ordinary reduction then so does  $X \times_K X$ . Now the assertion follows from Conjecture 3.10.  $\square$

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